q-ANALOGUE OF THE BEREZIN QUANTIZATION METHOD

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1 Introduction

The concept of deformation quantization of a symmetric manifold M has been defined by Bayen, Flato, Fronsdal, Lichnerovich, and Sternheimer in [1]. Deformation quantization means a formal $*$-product

$$f_1 * f_2 = f_1 \cdot f_2 + \sum_{k=1}^{\infty} C_k(f_1, f_2) t^k, \qquad f_1, f_2 \in C^\infty(M)$$

with some additional properties, where $C_k : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bidifferential operators.

In the special case of the unit disc in $\mathbb{C}$ with $SU_{1,1}$-invariant symplectic structure, a formal $*$-product and explicit fopmulae for C_k, $k \in \mathbb{N}$, are derivable by a method of Berezin [4, 2].

Our intention is to replace the ordinary disc with its q-analogue. We are going to produce $U_q\mathfrak{su}_{11}$-invariant formal deformation of our quantum disc and to obtain an explicit formula for C_k, $k \in \mathbb{N}$, using a q-analogue of the Berezin method [10].

Our work is closely related to the paper of Klimek and Lesniewski [5] on two-parameter deformation of the unit disc. The explicit formulae for C_k we provide below work as a natural complement to the results of this paper.

2 Covariant symbols of linear operators

Everywhere in the sequel the field of complex numbers $\mathbb{C}$ is assumed as a ground field. Let also $q \in (0, 1)$.

Consider the well known algebra $\mathrm{Pol}(\mathbb{C})_q$ with two generators z, z^* and a single commutation relation $z^* z = q^2 z z^* + 1 - q^2$. Our intention is to produce a formal $*$-product

$$f_1 * f_2 = f_1 \cdot f_2 + \sum_{k=1}^{\infty} C_k(f_1, f_2) t^k, \qquad f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q, \tag{2.1}$$

(with some remarkable properties) to be given by explicit formulae for bilinear operators $C_k : \mathrm{Pol}(\mathbb{C})_q \times \mathrm{Pol}(\mathbb{C})_q \to \mathrm{Pol}(\mathbb{C})_q$.

We describe in this section the method of producing this $*$-product whose idea is due to F. Berezin.

It was explained in [8] that the vector space $D(\mathbb{U})'_q$ of formal series $\sum\limits_{j,k=0}^{\infty} a_{jk} z^j z^{*k}$ with complex coefficients is a q-analogue of the space of distributions in the unit disc $\mathbb{U} = \{z \in \mathbb{C} | \, |z| < 1\}$. Equip this space of formal series with the topology of coefficientwise convergence. Since $\{z^j z^{*k}\}_{j,k \in \mathbb{Z}_+}$ constitute a basis in the vector space $\mathrm{Pol}(\mathbb{C})_q$, $\mathrm{Pol}(\mathbb{C})_q$ admits an embedding into $D(\mathbb{U})'_q$ as a dense linear subvariety.

Consider the unital subalgebra $\mathbb{C}[z]_q \subset \mathrm{Pol}(\mathbb{C})_q$ generated by $z \in \mathrm{Pol}(\mathbb{C})_q$. Let $\alpha > 0$. We follow [5] in equipping the vector space $\mathbb{C}[z]_q$ with the scalar product $(z^j, z^k)_\alpha = \delta_{jk} \dfrac{(q^2; q^2)_k}{(q^{4\alpha+2}; q^2)_k}$, $j, k \in \mathbb{Z}_+$, where $(a; q^2)_k = (1-a)(1-q^2 a) \ldots (1-q^{2(k-1)} a)$. Let $L^2_a(d\nu_\alpha)_q$ be the a completion of $\mathbb{C}[z]_q$ with respect to the norm $\|\psi\|_\alpha = (\psi, \psi)_\alpha^{1/2}$. It was demonstrated in [5] that the Hilbert space $L^2_a(d\nu_\alpha)_q$ is a q-analogue of the weighted Bergman space. Let $\widehat{z}$ be a linear operator of multiplication by z:

$$\widehat{z} : L^2_a(d\nu_\alpha)_q \to L^2_a(d\nu_\alpha)_q; \qquad \widehat{z} : \psi(z) \mapsto z \cdot \psi(z),$$

and denote by $\widehat{z}^*$ the adjoint operator in $L^2_a(d\nu_\alpha)_q$ to $\widehat{z}$. The definition of the scalar product in $L^2_a(d\nu_\alpha)_q$ implies that the operators $\widehat{z}$, $\widehat{z}^*$ are bounded. Equip the space $\mathcal{L}_\alpha$ of bounded linear operators in $L^2_a(d\nu_\alpha)_q$ with the weakest topology in which all the linear functionals

$$l_{\psi_1,\psi_2} : \mathcal{L}_\alpha \to \mathbb{C}, \qquad l_{\psi_1,\psi_2} : A \mapsto (A\psi_1, \psi_2)_\alpha, \qquad \psi_1, \psi_2 \in \mathbb{C}[z]_q$$

are continuous. The following proposition is a straightforward consequence of the definitions (see the proof in [8]).

Proposition 2.1 *Given any bounded linear operator $\widehat{f}$ in the Hilbert space $L^2_a(d\nu_\alpha)_q$, there exists a unique formal series $f = \sum\limits_{j,k=0}^{\infty} a_{jk} z^j z^{*k} \in D(\mathbb{U})'_q$ such that $\widehat{f} = \sum\limits_{j,k=0}^{\infty} a_{jk} \widehat{z}^j \widehat{z}^{*k}$.*

Thus we get an injective linear map $\mathcal{L}_\alpha \to D(\mathbb{U})'_q$, $\widehat{f} \mapsto f$. The distribution f is called a *covariant symbol* of the linear operator $\widehat{f}$.

Remark 2.2. For an arbitrary $f \in \mathrm{Pol}(\mathbb{C})_q$, there exists a unique operator $\widehat{f} \in \mathcal{L}_\alpha$ with the covariant symbol f. Specifically, for $f = \sum\limits_{j,k=0}^{N(f)} a_{jk} z^j z^{*k}$, one has $\widehat{f} = \sum\limits_{j,k=0}^{N(f)} a_{jk} \widehat{z}^j \widehat{z}^{*k}$.

We follow F. Berezin in producing the $*$-product of covariant symbols using the ordinary product of the associated linear operators.

Let $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$ and $\widehat{f_1}, \widehat{f_2} \in \mathcal{L}_\alpha$ be the operators whose covariant symbols are f_1, f_2. Under the notation $t = q^{4\alpha}$, let $m_t(f_1, f_2)$ stand for the covariant symbol of the product $\widehat{f_1} \cdot \widehat{f_2}$ of the linear maps $\widehat{f_1}$, $\widehat{f_2}$. Evidently, we have constructed a bilinear map $m_t : \mathrm{Pol}(\mathbb{C})_q \times \mathrm{Pol}(\mathbb{C})_q \to D(\mathbb{U})'_q$.

The $*$-product $f_1 * f_2$ of $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$ is to be introduced by replacement of the one-parameter family of distributions $m_t(f_1, f_2)$, $t \in (0,1)$, with its asymptotic expansion as $t \to 0$.

3 $*$-Product

The term *'order one differential calculus over the algebra* $\mathrm{Pol}(\mathbb{C})_q$*'* stand for a $\mathrm{Pol}(\mathbb{C})_q$-bimodule $\Omega^1(\mathbb{C})_q$ equipped with a linear map $d : \mathrm{Pol}(\mathbb{C})_q \to \Omega^1(\mathbb{C})_q$ such that
i) d satisfies the Leibniz rule $d(f_1 f_2) = df_1 \cdot f_2 + f_1 \cdot df_2$ for any $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$,
ii) $\Omega^1(\mathbb{C})_q$ is a linear span of $f_1 \cdot df_2 \cdot f_3$, $f_1, f_2, f_3 \in \mathrm{Pol}(\mathbb{C})_q$ (see [6]).

One can find in [11] a construction of that kind of order one differential calculus for a wide class of prehomogeneous vector spaces V. In the case $V = \mathbb{C}$ we deal with this calculus is well known; it can be described in terms of the following commutation relations:

$$z \cdot dz = q^{-2} dz \cdot z, \qquad z^* dz^* = q^2 dz^* z^*, \qquad z^* dz = q^2 dz \cdot z^*, \qquad z \cdot dz^* = q^{-2} dz^* z.$$

The partial derivatives $\frac{\partial^{(r)}}{\partial z}, \frac{\partial^{(r)}}{\partial z^*}, \frac{\partial^{(l)}}{\partial z}, \frac{\partial^{(l)}}{\partial z^*}$ are linear operators in $\mathrm{Pol}(\mathbb{C})_q$ given by

$$df = \frac{\partial^{(r)} f}{\partial z} dz + \frac{\partial^{(r)} f}{\partial z^*} dz^* = dz \frac{\partial^{(l)} f}{\partial z} + dz^* \frac{\partial^{(l)} f}{\partial z^*},$$

with $f \in \mathrm{Pol}(\mathbb{C})_q$.

Let $\widetilde{\Box} : \mathrm{Pol}(\mathbb{C})_q^{\otimes 2} \to \mathrm{Pol}(\mathbb{C})_q^{\otimes 2}$, $m_0 : \mathrm{Pol}(\mathbb{C})_q^{\otimes 2} \to \mathrm{Pol}(\mathbb{C})_q$ be linear operators given by

$$\widetilde{\Box}(f_1 \otimes f_2) = \left(\frac{\partial^{(r)} f_1}{\partial z^*} \otimes 1 \right) \cdot q^{-2}(1 - (1 + q^{-2}) z^* \otimes z + q^{-2} z^{*2} \otimes z^2) \cdot \left(1 \otimes \frac{\partial^{(l)} f_2}{\partial z} \right),$$

$m_0(f_1 \otimes f_2) = f_1 f_2$, with $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$.

Theorem 3.1 *For all $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$, the following asymptotic expansion in $D(\mathbb{U})'_q$ is valid:*

$$m_t(f_1, f_2) \underset{t \to 0}{\simeq} f_1 * f_2, \qquad \text{with}$$

$$f_1 * f_2 = f_1 \cdot f_2 + \sum_{k=1}^{\infty} C_k(f_1, f_2) t^k \in \mathrm{Pol}(\mathbb{C})_q[[t]], \tag{3.1}$$

$$C_k(f_1, f_2) = m_0 \left(\left(p_k \left(\widetilde{\Box} \right) - p_{k-1} \left(\widetilde{\Box} \right) \right) (f_1 \otimes f_2) \right), \tag{3.2}$$

and $p_k(x)$, $k \in \mathbb{Z}_+$, are polynomials given by

$$p_k(x) = \sum_{j=0}^{k} \frac{(q^{-2k}; q^2)_j}{(q^2; q^2)_j^2} q^{2j} \prod_{i=0}^{j-1} (1 - q^{2i}((1 - q^2)^2 x + 1 + q^2) + q^{4i+2}). \tag{3.3}$$

This statement is to be proved in the next section, using the results of [10] on a q-analogue of the Berezin transform [12].

We are grateful to H. T. Koelink who attracted our attention to the fact that the polynomials $p_k(x)$ differ from the polynomials of Al-Salam – Chihara [7] only by normalizing multiples and a linear change of the variable x.

4 A q-analogue of the Berezin transform

Remind the notation $t = q^{4\alpha}$, with $q \in (0,1)$, $\alpha > 0$.

Consider the linear map $\mathrm{Pol}(\mathbb{C})_q \to \mathcal{L}_\alpha$ which sends a polynomial $\overset{\circ}{f} = \sum\limits_{jk} b_{jk} z^{*j} z^k$ to the linear operator $\widehat{f} = \sum\limits_{jk} b_{jk} \widehat{z}^{*j} \widehat{z}^k$. The polynomial $\overset{\circ}{f}$ will be called a *contravariant symbol* of the linear operator $\widehat{f}$.

Note that our definitions of covariant and contravariant symbols agree with the conventional ones, as one can observe from [10] (specifically, see proposition 6.6 and lemma 7.2 of that work).

The term *'q-transform of Berezin'* will be stand for the linear operator $B_{q,t}$: $\mathrm{Pol}(\mathbb{C})_q \to D(\mathbb{U})'_q$, $B_{q,t} : \overset{\circ}{f} \mapsto f$, which sends the contravariant symbols of linear operators $\widehat{f} = \sum\limits_{jk} b_{jk} \widehat{z}^{*j} \widehat{z}^k$ to their covariant symbols.

REMARK 4.1. It is easy to extend the operators $B_{q,t}$ onto the entire *'space of bounded functions in the quantum disc'* via a non-standard approach to their construction (see [10]).

[10, proposition 5.5] imply

Proposition 4.1 *Given arbitrary* $\overset{\circ}{f} \in \mathrm{Pol}(\mathbb{C})_q$, *the following asymptotic expansion in the topological vector space* $D(\mathbb{U})'_q$ *is valid:*

$$B_{q,t} \overset{\circ}{f} \underset{t \to 0}{\simeq} \overset{\circ}{f} + \sum_{k=1}^{\infty} ((p_k(\square) \overset{\circ}{f} - p_{k-1}(\square) \overset{\circ}{f}) t^k,$$

with $\square$ *being a q-analogue of the Laplace-Beltrami operator*

$$\square f \overset{\mathrm{def}}{=} (1 - zz^*)^2 \frac{\partial^{(l)}}{\partial z^*} \frac{\partial^{(l)} f}{\partial z} = q^2 \frac{\partial^{(r)}}{\partial z^*} \frac{\partial^{(r)} f}{\partial z} (1 - zz^*)^2,$$

with $f \in \mathrm{Pol}(\mathbb{C})_q$ *and* p_k, $k \in \mathbb{Z}_+$, *being polynomials given by (3.3).*

It follows from the definition of the bilinear maps m_t, $t \in (0,1)$, that for all $i, j, k, l \in \mathbb{Z}_+$, $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$,

$$m_t(z^i f_1, f_2) = z^i m_t(f_1, f_2),$$
$$m_t(f_1, f_2 z^{*l}) = m_t(f_1, f_2) z^{*l},$$
$$m_t(z^{*j}, z^k) = B_{q,t}(z^{*j} z^k).$$

Hence for all $i,j,k,l\in\mathbb{Z}_+$ one has

$$m_t((z^iz^{*j}),(z^kz^{*l}))=z^iB_{q,t}(z^{*j}z^k)z^{*l}. \tag{4.1}$$

We are about to deduce theorem 3.1 from (4.1) and proposition 4.1. In fact, one can easily demonstrate as in [10, proposition 8.3] that

$$\Box(f_2(z^*)\cdot f_1(z))=q^2\frac{\partial^{(r)}f_2(z^*)}{\partial z^*}(1-zz^*)^2\frac{\partial^{(l)}f_1(z)}{\partial z}=$$
$$=\frac{\partial^{(r)}f_2(z^*)}{\partial z^*}q^{-2}(1-(1+q^{-2})z^*z+q^{-2}z^{*2}z^2)\frac{\partial^{(l)}f_1(z)}{\partial z}$$

for arbitrary polynomials $f_1(z)$, $f_2(z^*)$. What remains is to compare this expression for $\Box$ with the definition of $\widetilde{\Box}$ and apply the fact that $\{z^iz^{*j}\}_{i,j\in\mathbb{Z}_+}$ constitute a basis in the vector space $\mathrm{Pol}(\mathbb{C})_q$.

5 A formal associativity

Proposition 5.1 *The multiplication in* $\mathrm{Pol}(\mathbb{C})_q[[t]]$ *given by the bilinear map*

$$m:\mathrm{Pol}(\mathbb{C})_q[[t]]\times\mathrm{Pol}(\mathbb{C})_q[[t]]\to\mathrm{Pol}(\mathbb{C})_q[[t]],$$

$$m:\sum_{j=0}^{\infty}a_jt^j\times\sum_{k=0}^{\infty}b_kt^k\mapsto\sum_{i=0}^{\infty}\left(\sum_{j+k=i}a_j*b_k\right)t^i\ {}^1, \tag{5.1}$$

with $\{a_j\}_{j\in\mathbb{Z}_+},\{b_k\}_{k\in\mathbb{Z}_+}\in\mathrm{Pol}(\mathbb{C})_q$, *is associative.*

Proof. Introduce the algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)$ of all linear operators in the vector space $\mathbb{C}[z]_q$, and the algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)[[t]]$ of formal series with coefficients in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)$. To prove our statement, it suffices to establish an isomorphism of the algebra $\mathrm{Pol}(\mathbb{C})_q[[t]]$ equipped with the multiplication m and a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)[[t]]$ given the standard multiplication. Let $\mathcal{I}:\mathrm{Pol}(\mathbb{C})_q\to\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)[[t]]$ be such a linear operator that for all $j,k,m\in\mathbb{Z}_+$

$$\mathcal{I}(z^jz^{*k}):z^m\mapsto\begin{cases}\dfrac{(q^{2m};q^{-2})_k}{(tq^{2m};q^{-2})_k}z^{m-k+j} & ,\quad k\le m\\ \qquad 0 & ,\quad k>m\end{cases}.$$

(More precisely, one should replace the rational function $1/(tq^{2m};q^{-2})_k$ of an indeterminate t with its Teylor expansion.)

The following lemma follows from the construction of [10, section 7].

[1] The outward sum clearly converges in the topological vector space $\mathrm{Pol}(\mathbb{C})_q[[t]]$.

Lemma 5.2 *The linear map*

$$Q : \mathrm{Pol}(\mathbb{C})_q[[t]] \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)[[t]],$$

$$Q : \sum_{j=0}^{\infty} f_j t^j \mapsto \sum_{j=0}^{\infty} \mathcal{I}(f_j) t^j\ {}^2, \qquad \{f_j\}_{j\in\mathbb{Z}_+} \subset \mathrm{Pol}(\mathbb{C})_q,$$

is injective, and for all $\psi_1, \psi_2 \in \mathrm{Pol}(\mathbb{C})_q[[t]]$ *one has* $Qm(\psi_1,\psi_2) = (Q\psi_1)\cdot(Q\psi_2)$.

Lemma 5.2 implies the associativity of the multiplication m in $\mathrm{Pol}(\mathbb{C})_q[[t]]$. Thus, proposition 5.1 is proved. ■

Define a linear operator $*$ in $\mathrm{Pol}(\mathbb{C})_q[[t]]$ by

$$\left(\sum_{j=0}^{\infty} f_j t^j\right)^* = \sum_{j=0}^{\infty} f_j^* t^j, \qquad \{f_j\}_{j\in\mathbb{Z}_+} \subset \mathrm{Pol}(\mathbb{C})_q.$$

Proposition 5.3 $*$ *is an involution in* $\mathrm{Pol}(\mathbb{C})_q[[t]]$ *equipped by* m *as a multiplication:*

$$m(\psi_1,\psi_2)^* = m(\psi_2^*,\psi_1^*), \qquad \psi_1,\psi_2 \in \mathrm{Pol}(\mathbb{C})_q[[t]].$$

Proof. For all $f_1, f_2 \in \mathrm{Pol}(\mathbb{C})_q$ one has

$$(m_0(f_1\otimes f_2))^* = m_0(f_2^*\otimes f_1^*),$$

$$\widetilde{\square}^{21}(f_1\otimes f_2)^{*\otimes *} = \widetilde{\square}(f_1^*\otimes f_2^*)$$

with $\widetilde{\square}^{21} = c_0 \square c_0$, and c_0 being the flip of tensor multiples. What remains is to observe that the coefficients of $p_n(x)$, $n\in\mathbb{Z}_+$, are real, and to apply (5.1), (3.1), (3.2). ■

6 $U_q\mathfrak{su}_{1,1}$-invariance

Remind some well known results on the quantum group $SU_{1,1}$ and the quantum disc (see, for example, [3, 9]).

The quantum universal enveloping algebra $U_q\mathfrak{sl}_2$ is a Hopf algebra over $\mathbb{C}$ determined by the generators K, K^{-1}, E, F, and the relations

$$KK^{-1} = K^{-1}K = 1, \qquad K^{\pm1}E = q^{\pm2}EK^{\pm1}, \qquad K^{\pm1}F = q^{\mp2}FK^{\pm1},$$

$$EF - FE = (K-K^{-1})/(q-q^{-1}).$$

Comultiplication $\Delta : U_q\mathfrak{sl}_2 \to U_q\mathfrak{sl}_2 \otimes U_q\mathfrak{sl}_2$, counit $\varepsilon : U_q\mathfrak{sl}_2 \to \mathbb{C}$ and antipode $S : U_q\mathfrak{sl}_2 \to U_q\mathfrak{sl}_2$ are given by

$$\Delta(K^{\pm1}) = K^{\pm1}\otimes K^{\pm1}, \qquad \Delta(E) = E\otimes 1 + K\otimes E, \qquad \Delta(F) = F\otimes K^{-1} + 1\otimes F,$$

$$\varepsilon(E) = \varepsilon(F) = \varepsilon(K^{\pm1}-1) = 0,$$

[2] The convergence of the series $\sum_{j=0}^{\infty}\mathcal{I}(f_j)t^j$ in the space $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[z]_q)[[t]]$ is obvious.

$$S(K^{\pm 1}) = K^{\mp 1}, \qquad S(E) = -K^{-1}E, \qquad S(F) = -FK.$$

The structure of Hopf algebra allows one to define a tensor product of $U_q\mathfrak{sl}_2$-modules and a tensor product of their morphisms. Thus, we obtain a tensor category of $U_q\mathfrak{sl}_2$-modules.

Consider an algebra F equipped also with a structure of $U_q\mathfrak{sl}_2$-module. F is called a $U_q\mathfrak{sl}_2$-module algebra if the multiplication

$$m_F : F \otimes F \to F, \qquad m_F : f_1 \otimes f_2 \mapsto f_1 f_2, \qquad f_1, f_2 \in F,$$

is a morphism of $U_q\mathfrak{sl}_2$-modules. (In the case F has a unit, the above definition should also include its invariance: $\xi \cdot 1 = \varepsilon(\xi)1$, $\xi \in U_q\mathfrak{sl}_2$).

The following relations determine a structure of $U_q\mathfrak{sl}_2$-module algebra on $\mathbb{C}[z]_q$:

$$K^{\pm 1} z = q^{\pm 2} z, \qquad Fz = q^{1/2}, \qquad Ez = -q^{1/2} z^2. \tag{6.1}$$

Equip $U_q\mathfrak{sl}_2$ with an involution:

$$E^* = -KF, \qquad F^* = -EK^{-1}, \qquad (K^{\pm 1})^* = K^{\pm 1},$$

and let $U_q\mathfrak{su}_{1,1}$ stand for the Hopf $*$-algebra produced this way. An involutive algebra F is said to be $U_q\mathfrak{su}_{1,1}$-module algebra if it is $U_q\mathfrak{sl}_2$-module algebra, and the involutions in F and $U_q\mathfrak{su}_{1,1}$ agree as follows:

$$(\xi f)^* = (S(\xi))^* f^*, \qquad \xi \in U_q\mathfrak{su}_{1,1},\ f \in F. \tag{6.2}$$

(6.1) determines a structure of $U_q\mathfrak{su}_{1,1}$-module algebra in $\mathrm{Pol}(\mathbb{C})_q$. Thus, each of the vector spaces $\mathrm{Pol}(\mathbb{C})_q$, $\mathrm{Pol}(\mathbb{C})_q[[t]]$ is equipped with a structure of $U_q\mathfrak{su}_{1,1}$-module.

Proposition 6.1 $\mathrm{Pol}(\mathbb{C})_q[[t]]$ *with the multiplication defined above and the involution* $*$ *is a* $U_q\mathfrak{su}_{1,1}$*-module algebra.*

Proof. Since $\mathrm{Pol}(\mathbb{C})_q$ is a $U_q\mathfrak{su}_{1,1}$-module algebra, (6.2) is valid for $F = \mathrm{Pol}(\mathbb{C})_q$. Hence it is also true for $F = \mathrm{Pol}(\mathbb{C})_q[[t]]$. What remains is to prove that $\mathrm{Pol}(\mathbb{C})_q[[t]]$ is a $U_q\mathfrak{sl}_2$-module algebra. For that, by a virtue of (3.1), (3.2), it suffices to demonstrate that the linear maps m_0 and $\widetilde{\Box}$ are morphisms of $U_q\mathfrak{sl}_2$-modules. As for m_0, this property has already been mentioned. So we need only to consider $\widetilde{\Box}$. Given any polynomials $f_1(z^*)$, $f_2(z)$, it follows from $\Box(f_1(z^*)f_2(z)) = \sum\limits_{jk} b_{jk} z^{*j} z^k$, $b_{jk} \in \mathbb{C}$, that $\widetilde{\Box}(f_1(z^*) \otimes f_2(z)) = \sum\limits_{jk} b_{jk} z^{*j} \otimes z^k$, and

$$\widetilde{\Box}(g_1(z)f_1(z^*) \otimes f_2(z)g_2(z^*)) = (g_1(z) \otimes 1)\widetilde{\Box}(f_1(z^*) \otimes f_2(z))(1 \otimes g_2(z^*)).$$

Thus, it suffices to prove that $\Box$ is a morphism $U_q\mathfrak{sl}_2$-modules. This latter result is obtained in [9] (It is a consequence of $U_q\mathfrak{su}_{1,1}$-invariance of the differential calculus in the quantum disc considered there). ■

REMARK 6.2. The works [8, 9] deal with the $U_q\mathfrak{su}_{1,1}$-module algebra $D(\mathbb{U})_q$ of 'finite functions in the quantum disc'. (The space $D(\mathbb{U})'_q$ mentioned in this work is dual to $D(\mathbb{U})_q$). The relations (3.1) – (3.3) determine a formal deformation of $D(\mathbb{U})_q$ in the class of $U_q\mathfrak{su}_{1,1}$-module algebras, that is, it allows one to equip $D(\mathbb{U})_q[[t]]$ with a structure of $U_q\mathfrak{su}_{1,1}$-module algebra over the ring $\mathbb{C}[[t]]$.

7 Concluding notes

We have demonstrated that the method of Berezin allows one to produce a formal deformation for a q-analogue of the unit disc. In [11], q-analogues for arbitrary bounded symmetric domains were constructed. We hope in that essentially more general setting, the method of Berezin will help remarkable results to be obtained.

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